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Boundary conditions and the first-order conserved density for gas dynamics

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Abstract. Boundary conditions for gas dynamics are chosen to make the first-order conserved density finitely integrable. The action of the corresponding third-order Noetherian symmetry on the boundary conditions is discussed. A generalisation to first-order diagonalised quasilinear systems is considered.

1. Introduction

The equations of gas dynamics in one space dimension are

$$u_t + uu_x + f(v)v_x = 0$$

$$v_t + vu_x + uv_x = 0$$

where u is the velocity and v is the density of the gas. The positive function f is related to the pressure $p(v)$ by $p'(v) = vf(v)$. Verosky (1984) showed that for an ideal gas pressure $p(v) = v^\gamma$, $\gamma \neq 1$, there is a unique first-order conserved density but for arbitrary pressure the first-order density

$$T = \frac{v_x}{u_x^2 - (f(v)/v)v_x^2}$$

is conserved, at least in the sense of the formal variational calculus, which means that there is a corresponding flux X such that $D_tT = D_xX$ on solutions of the gas dynamics equations. The derivatives D_t and D_x are total derivatives and include t and x dependencies as they may occur through intermediate functions such as u, v, u_x and v_X. The flux for T given above is

$$X = \frac{uv_x - vu_x}{u_x^2 - (f(v)/v)v_x^2}$$

and it is a simple exercise in differentiation to check that $D_tT = D_xX$.

If T and X are polynomial functions of the dependent variables and their derivatives and the boundary conditions require that the solutions vanish rapidly at infinity then a formal conserved density is an actual conserved density because

$$\frac{d}{dt}\int_{-\infty}^{\infty} T\, dx = \int_{-\infty}^{\infty} D_tT\, dx = \int_{-\infty}^{\infty} D_xX\, dx = X|_{-\infty}^{\infty} = 0$$

and the integrated density is finite and constant in time. See Olver (1986) for several examples of this, such as the conserved densities of the Korteweg-de Vries equation.

Since for gas dynamics the density T is a rational function, suitable boundary conditions must be provided to ensure that the integral of T is finite and constant in time. This is the main task of this paper.

2. The boundary conditions

The first condition is quite natural for gas dynamics.

(1) For all $t \geqslant 0$ the density v rapidly approaches positive constants v_1 and v_2 as x approaches $-\infty$ and $+\infty$ respectively. This means that all derivatives of v approach zero at $\pm\infty$ and hence we must have the second condition.

(2) For all $t \geqslant 0$ the velocity u approaches solutions of Hopf's equation

$$u_t + uu_x = 0$$

at $\pm\infty$. (This equation has many names (Riemann's equation, inviscid Burger's equation, for example) but we will simply call it Hopf's equation.) The flux thus approaches some constant multiplying u_x^{-1} at $\pm\infty$. To have a true conserved density, this must approach zero. This leads to the third condition.

(3) For all $t \geqslant 0$ the derivative u_x approaches ∞ at $\pm\infty$. Note that this condition is compatible with condition (2) and that no shocks will form when u_x is positive as shown by a simple look at the solution of Hopf's equation in terms of characteristics. The values of u_x for all x and all $t \geqslant 0$ are discussed below.

(4) The final condition is one on the initial values of u and v and not a boundary condition. The denominator satisfies $u_x^2 - (f(v)/v)v_x^2 > 0$ for the initial values. This automatically implies that $u_x > 0$ for all x at $t = 0$.

Note that condition (4) implies that T is always finite and together with (1) means that T is finitely integrable. Also, this condition continues to hold even as u and v evolve and T will remain integrable. A rough sketch of how this may be proved now follows. Let $p = u + g(v)$ and $q = u - g(v)$ (where $g'(v) = (f(v)/v)^{1/2}$) be characteristic coordinates. Then condition (4) says that $p_x q_x > 0$ for all x at $t = 0$. But by conditions (1) and (3) p_x and q_x go to *positive* ∞ as x approaches ∞. Thus p_x and q_x are both positive for all x at $t = 0$. This implies p and q are increasing functions. But p and q evolve according to diagonalised equations

$$p_t + A(p, q)p_x = 0 \qquad q_t + B(p, q)q_x = 0$$

where $A = u + (vf(v))^{1/2}$ and $B = u - (vf(v))^{1/2}$ in terms of u and v which, in turn, are functions of p and q. This means that p and q are constants on the flows of characteristic curves

$$\frac{dx}{dt} = A(p, q) \qquad \text{and} \qquad \frac{dx}{dt} = B(p, q)$$

respectively. But these flows cannot change the qualitative shapes of the graphs of p and q as increasing functions of x and hence $u_x^2 - (f(v)/v)v_x^2 = p_x q_x$ remains positive as u and v evolve. This argument continues to hold as long as the functions A and B are such that the above differential equations admit unique solutions for any initial values of x and thus induce homeomorphisms among the lines $t = $ constant. A similar argument works for any diagonalised first-order system of PDE (into which gas dynamics may be transformed) with monotone initial values as long as the corresponding ODE are well behaved. It is not even necessary in the case of gas dynamics to require that v be positive to prevent $A = B$, for it is still true that $p_x q_x = u_x^2 > 0$ for zero v.

These four conditions imply that T is a well behaved conserved density whose integral is finite and constant in time. The conditions are not unphysical because they have a simple physical interpretation: a tube of expanding gas, with expansion velocities tending to infinity at the ends of the tube. The density of the gas approaches constants at the two ends. Even the density T can be interpreted as a measure of 'simplicity'. A simple wave is one where u and v are functionally related

$$u = F(w(x, t)) \qquad \text{and} \qquad v = G(w(x, t))$$

which happens if and only if either v or the denominator of T vanishes. To see this consider the Jacobian

$$u_t v_x - v_t u_x = -(uu_x + fv_x)v_x + (vu_x + uv_x)u_x = v(u_x^2 - (f(v)/v)v_x^2).$$

Thus the 'simplicity' that occurs when v is zero will not cause the denominator of T to vanish. Solutions u and v become 'more simple' when the denominator becomes smaller or when T becomes larger relative to vv_x. Thus the integral of T is a measure of how close relative to vv_x to a simple wave the solution is. This quantity remains constant in time.

3. Higher-order symmetries

Gas dynamics has a Hamiltonian structure which can be used to produce a third-order symmetry from the conserved density T by Noether's theorem. A third-order symmetry is a flow in the space of solutions for the gas dynamics equations given by an evolution equation of third order in the space derivatives of u and v. Again see Olver (1986) for a definitive study of higher-order symmetries. In this case it is

$$\begin{pmatrix} u \\ v \end{pmatrix}_s = -\begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix} \begin{pmatrix} \partial/\partial u - D_x\, \partial/\partial u_x \\ \partial/\partial v - D_x\, \partial/\partial v_x \end{pmatrix} \frac{v_x}{u_x^2 - (f(v)/v)v_x^2}$$

where s is the parameter of this flow. Note the expanded forms of the variational derivatives. Two total derivatives D_x are eventually taken of first-order quantities leading to a third-order right-hand side. Note that only powers of the denominator $u_x^2 - (f(v)/v)v_x^2$ will occur in the denominators of the terms in the right-hand side (by the quotient rule of elementary calculus), so that the evolution equation is well defined on our restricted space of solutions, at least for a short time since that denominator starts out as a positive quantity everywhere.

More importantly, note that the evolution of v is governed by

$$v_s = -D_x^2 \frac{2u_x v_x}{(\text{denom})^2}$$

and that if $v \equiv$ constant then $v_s \equiv 0$. This means that this symmetry will not change the boundary conditions on v. Also, for $v \equiv$ constant, the symmetry reduces to the third-order symmetry of Hopf's equation

$$u_s = -D_x^2(1/u_x^2).$$

Thus both boundary conditions are preserved by this third-order symmetry. The integral of T will of course remain constant since this symmetry is a Hamiltonian system with

Hamiltonian function T and Hamiltonian systems always conserve their own Hamiltonian density. This property of preserving both boundary conditions might not in general be possessed by a higher-order symmetry

$$u_s = P \qquad v_s = Q$$

because, even if the initial value $v \equiv$ constant at $t=0$ holds, then Q, which depends possibly on u and its x derivatives, may not be zero and the symmetry may change v to a non-constant function. Only if $Q=0$ for $v \equiv$ constant, as is the case for the above third-order equation, will the property hold. Every term of Q would have to have a derivative of v in the numerator.

As a negative example consider a non-linear Hamiltonian structure for the shallow-water wave equations (gas dynamics with $f(v)=1$):

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = -\begin{pmatrix} D_x & \frac{1}{2}D_x u \\ \frac{1}{2}uD_x & \frac{1}{2}(vD_x + D_x v) \end{pmatrix}\begin{pmatrix} \partial/\partial u \\ \partial/\partial v \end{pmatrix} uv.$$

Such alternative Hamiltonian structures were first discovered by Nutku (1987) and show that the gas dynamics equations are biHamiltonian, the fascinating implications of which were first explored by Magri (1978). According to Noether's theorem

$$\begin{pmatrix} u \\ v \end{pmatrix}_s = -\begin{pmatrix} D_x & \frac{1}{2}D_x u \\ \frac{1}{2}uD_x & \frac{1}{2}(vD_x + D_x v) \end{pmatrix}\begin{pmatrix} \partial/\partial u - D_x\, \partial/\partial u_x \\ \partial/\partial u - D_x\, \partial/\partial u_x \end{pmatrix} \frac{v_x}{u_x^2 - (1/v)v_x^2}$$

is a third-order symmetry, but it reduces to

$$\begin{pmatrix} u \\ v \end{pmatrix}_s = -\begin{pmatrix} D_x & \frac{1}{2}D_x u \\ \frac{1}{2}u_x D & cD_x \end{pmatrix}\begin{pmatrix} 0 \\ -D_x u_x^{-2} \end{pmatrix}$$

at $t=0$ if v is a constant c. In particular

$$v_s = cD_x^2 u_x^{-2}$$

so that v would have to be zero at $\pm\infty$ for this third-order symmetry to preserve the boundary condition on v.

4. Diagonalised systems

The system

$$u_t^k = A^k(u_1, \ldots, u_n)u_x^k \qquad k = 1, \ldots, n$$

for n dependent variables $u^1, \ldots, u^n$ has, for special choices of A^k, a first-order conserved density of the form

$$T = \frac{B^1}{u_x^1} + \frac{B^2}{u_x^2} + \ldots + \frac{B^n}{u_x^n}$$

where the B^k are functions of the u^k. The flux is of the form

$$X = \frac{A^1B^1}{u_x^1} + \frac{A^2B^2}{u_x^2} + \ldots + \frac{A^nB^n}{u_x^n}$$

and simply expanding $D_tT = D_xX$ and comparing coefficients of the first-order monomials gives conditions on the A^k for the existence of T. Gas dynamics can be written in this form, as we saw earlier. In fact one can include an entropy equation $u_t + us_x = 0$ and still diagonalise for a special choice of pressure $p(v, s)$ and get first-order conserved density as shown in Verosky (1986).

If T exists for a general diagonalised system then consider solutions where u_x^k is always positive for all k and x and where u_x^k approaches infinity rapidly enough at $\pm\infty$. Then the integral of T is finite and constant in time. The values of u^k are constant on the characteristic curves

$$\mathrm{d}x/\mathrm{d}t = A^k$$

and a graph that is everywhere increasing will not lose this topological property as it is transferred by the above flow in the xt plane. Thus the conditions on the u_x^k that keep T integrable persist at least until the above flows break down (shock formation, for example).

As it stands, the quasilinear system does not have an obvious Hamiltonian form but if it does, by Dubrovin and Novikov (1983) there is a change in the dependent variables that would allow it to be written in the form

$$\begin{pmatrix} v^1 \\ v^2 \\ \\ v^n \end{pmatrix}_t = \begin{pmatrix} D_x & & & & & 0 \\ & \ddots & & & & \\ & & D_x & & & \\ & & & -D_x & & \\ & & & & \ddots & \\ 0 & & & & & -D_x \end{pmatrix} \begin{pmatrix} \partial/\partial v^1 \\ \partial/\partial v^2 \\ \vdots \\ \partial/\partial v^n \end{pmatrix} H(v^1, v^2, \ldots, v^n).$$

The density T would become a new first-order rational conserved density and the conditions on the u_x^k would become new conditions that would keep T finitely integrable. The denominators in the new T would never be zero and the Noetherian third-order symmetry obtained by replacing the Hamiltonian H by T would only contain powers of the well behaved denominators of T and would be a well defined evolution equation for a short time at least.

5. Conclusion

The purpose of this paper has been to bridge the gap between the formal variational calculus which deals with higher-order conserved densities and conservation laws in a formal differential-algebraic manner and the actual analytic realities of these objects. In the case of gas dynamics, the rational first-order density is given a firm analytic footing by restricting the solutions to those satisfying certain reasonable conditions. The third-order symmetry resulting from Noether's theorem is also well behaved. Similar arguments hold for diagonalised quasilinear systems.

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